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Universal amplitude ratios and Coxeter geometry in the dilute A_L model

C. Korff[1] and K. A. Seaton[2]

[1] *C.N. Yang Institute for Theoretical Physics*
State University of New York at Stony Brook
Stony Brook, N.Y. 11794-3840, USA

[2] *School of Mathematical and Statistical Sciences*
La Trobe University, Victoria 3086, Australia

Abstract

The leading excitations of the dilute A_L model in regime 2 are considered using analytic arguments. The model can be identified with the integrable $\phi_{1,2}$ perturbation of the unitary minimal series $M_{L,L+1}$. It is demonstrated that the excitation spectrum of the transfer matrix satisfies the same functional equations in terms of elliptic functions as the exact S-matrices of the $\phi_{1,2}$ perturbation do in terms of trigonometric functions. In particular, the bootstrap equation corresponding to a self-fusing process is recovered. For the special cases $L = 3, 4, 6$ corresponding to the Ising model in a magnetic field, and the leading thermal perturbations of the tricritical Ising and three-state Potts model, as well as for the unrestricted model, $L = \infty$, we relate the structure of the Bethe roots to the Lie algebras $E_{8,7,6}$ and D_4 using Coxeter geometry. In these cases Coxeter geometry also allows for a single formula in generic Lie algebraic terms describing all four cases. For general L we calculate the spectral gaps associated with the leading excitation which allows us to compute universal amplitude ratios characteristic of the universality class. The ratios are of field theoretic importance as they enter the bulk vacuum expectation value of the energy momentum tensor associated with the corresponding integrable quantum field theories.

korff@insti.physics.sunysb.edu
k.seaton@latrobe.edu.au

1 Introduction

One of the most significant applications of two-dimensional integrable systems in field theory and statistical mechanics is to systems undergoing a continuous phase transition. Near the critical point the correlation length diverges rendering the system scale invariant. This leads to the well-known concept of universality classes, which describe the macroscopic scaling behavior of the thermodynamic quantities near the critical point. The universality classes can be identified with conformal field theories which encode the critical exponents in the spectrum of their primary fields. Besides the critical exponents, each class exhibits additional universal ratios of critical amplitudes, which might equally well serve as a characteristic, see e.g. [1] and references therein. In two dimensions exact non-perturbative results are available on these critical amplitudes, provided the off-critical model can effectively be described by an integrable perturbation of the conformal field theory governing the critical point. There are various ways to proceed, such as conformal perturbation theory, the bootstrap approach involving the construction of exact scattering matrices and the transfer matrix method. While we concentrate on the last of these in this article, it has intimate ties with the other techniques which we will point out in the course of our discussion.

We consider the integrable off-critical dilute A_L lattice model [2] which belongs to the class of interaction-round-a-face (IRF) models with adjacency matrix related to the simple Lie algebra A_L. The epithet "dilute" points out that neighbouring lattice sites are allowed to have the same height value in addition to those permitted in the conventional IRF models, where neighbouring heights take adjacent values in the Dynkin diagram. The dilute A_L model possesses four different branches [3] listed in Table 1. We focus in this article on regime 2 which can be associated with the integrable perturbation $\phi_{1,2}$ of the unitary minimal conformal models $M_{L,L+1}$. The Boltzmann weights in the off-critical region are expressed in Jacobi's theta functions depending on a spectral parameter u restricted to the interval listed in Table 1, and an elliptic nome $0 < |p| < 1$ which can be identified with the ordering field. The latter is of magnetic or thermal type depending on whether L is odd or even. As $p \to 0^{\pm}$ the system approaches criticality.

regime	interval	c	Δ^{-1}	λ
1	$0 < u < 3\lambda$	$1 - \frac{6}{(L+1)(L+2)}$	$4\frac{L+1}{L+4}$	$\frac{\pi}{4}\frac{L}{L+1}$
2	$0 < u < 3\lambda$	$1 - \frac{6}{L(L+1)}$	$4\frac{L+1}{L-2}$	$\frac{\pi}{4}\frac{L+2}{L+1}$
3	$3\lambda - \pi < u < 0$	$\frac{3}{2} - \frac{6}{(L+1)(L+2)}$	$4\frac{L+1}{L-2}$	$\frac{\pi}{4}\frac{L+2}{L+1}$
4	$3\lambda - \pi < u < 0$	$\frac{3}{2} - \frac{6}{L(L+1)}$	$4\frac{L+1}{L+4}$	$\frac{\pi}{4}\frac{L}{L+1}$

Table 1. The four branches of the dilute A models. Listed are the allowed values of the spectral and the crossing parameter entering the Boltzmann weights as well as the central charge and the anomalous scaling dimension of the perturbing operator.

In this article we put forward the leading off-critical excitations in the spectrum of the associated transfer matrix in regime 2 for arbitrary L when the thermodynamic

limit is taken. This allows us to determine the associated spectral gaps in the bulk limit, which determine the finite off-critical correlation length

$$\xi^{-1} = -\lim_{N\to\infty} \ln \frac{\Lambda_1}{\Lambda_0} \sim |p|^{\frac{1}{2-2\Delta}}\, \mathcal{S}^{\pm}(q'/|p|^{\frac{1-\Delta'}{1-\Delta}})^{-1} \;, \quad (p \to 0^{\pm}) \;. \tag{1}$$

Here Λ_1, Λ_0 are eigenvalues of the transfer matrix on a strip of length N corresponding to the lowest excited and the ground state, respectively. As the critical point is approached the spectral gap collapses, and the associated correlation length diverges with critical exponent determined by the anomalous scaling dimension $\Delta = \Delta_{1,2}$ of the perturbing field $\phi_{1,2}$. The renormalization group behaviour of the system is encoded in the bulk universal scaling functions $\mathcal{S}^{\pm}$ which depend on the dimensionless scaling variable $q'/|p|^{\frac{1-\Delta'}{1-\Delta}}$, with q' being a dummy variable for the perturbation by any other relevant field with scaling dimension Δ'. Further information on the renormalization group characteristics of the $\phi_{1,2}$ perturbation comes from the singular part f_s of the free energy density of the dilute A_L model [3, 4] and its associated bulk scaling functions $\mathcal{Q}^{\pm}$ as $p \to 0^{\pm}$,

$$f = -\lim_{N\to\infty} N^{-1} \ln \Lambda_0 = f_s + f_{ns} \;, \quad f_s \sim |p|^{\frac{1}{1-\Delta_{1,2}}}\, \mathcal{Q}^{\pm}(q'/|p|^{\frac{1-\Delta'}{1-\Delta}}). \tag{2}$$

The explicit form of the bulk scaling functions $\mathcal{S}^{\pm}$ and $\mathcal{Q}^{\pm}$ is not accessible via the dilute A_L model since it realizes perturbation by only one relevant field, i.e. $q' = 0$. However, besides the scaling dimension of the perturbing operator one can extract the following combination of critical amplitudes (see e.g. [1]):

$$\lim_{p\to 0^{\pm}} f_s \cdot \xi^2 = \mathcal{Q}^{\pm}(0)\mathcal{S}^{\pm}(0)^2 \;. \tag{3}$$

While each amplitude separately depends on the normalization of the fields and is a non-universal quantity, the above combination is universal in the sense that it only depends on the universality class determined by the minimal model $M_{L,L+1}$ and its $\phi_{1,2}$ perturbation. Since in the thermodynamic limit the statistical lattice model can be effectively described by an integrable quantum field theory the amplitude (3) is also accessible by exact S-matrix theory.

The best known example of the exact S-matrices corresponding to the $\phi_{1,2}$ perturbation of $M_{L,L+1}$ is Zamolodchikov's construction [5] related to the simple Lie algebra E_8 for the Ising model at $T = T_c$ in a magnetic field. In the context of the dilute A_L model this case corresponds to $L = 3$ in regime 2. Further examples exhibiting the exceptional algebraic structures E_7 and E_6 are the scattering matrices [6, 7, 8, 9] for $L = 4$ and $L = 6$ which are associated with the leading thermal perturbation of the tricritical Ising and tricritical three-state Potts model, respectively (see Table 2). For general L it has been proposed by Smirnov [10] that the $\phi_{2,1}$ and $\phi_{1,2}$ perturbations should be considered as reductions of the Zhiber-Mikhailov-Shabat (ZMS) model [11, 12]. While for arbitrary values of the purely imaginary coupling constant the theory violates unitarity, one can recover physically sensible theories at rational values by reducing the

state space. The corresponding scattering amplitudes are obtained at roots of unity, $q^L = 1$, from the Izergin-Korepin R-matrix [13] which can be associated with the quantum group $U_q(A_2^{(2)})$. The particle spectrum consists in general of a three-component kink and its bound states, called breathers. For $L = 3, 4, 6$ the aforementioned special cases are recovered with $L = 3$ being distinguished by the absence of any kink state [10].

L	conformal model	c	$\Delta_{1,2}^{-1}$	algebra
3	Ising in magnetic field	1/2	16	E_8
4	tricritical Ising	7/10	10	E_7
6	tricritical three-state Potts	6/7	7	E_6
∞	free boson	1	4	D_4

Table 2. Displayed are three special cases of minimal conformal models together with the values of the associated central charge and the anomalous scaling dimension of the perturbing operator. All of them are distinguished by an underlying Lie algebraic structure. In case of the unrestricted model, L=∞, one recovers the conformal theory associated with a free boson.

The universal quantity (3) can be extracted from the exact S-matrix theories by applying the thermodynamic Bethe ansatz [14, 15, 16, 17] which allows the two-particle scattering amplitude to be related to the infinite bulk vacuum energy:

$$\langle\Theta\rangle = \frac{2\pi m^2}{\delta_1} = -4\pi m^2 \mathcal{Q}^{\pm}(0)\mathcal{S}^{\pm}(0)^2, \quad -i\frac{d}{d\beta}\ln S(\beta) = -\sum_{n=1}^{\infty}\delta_n\, e^{-n|\beta|}\,. \tag{4}$$

Here $\langle\Theta\rangle$ denotes the infinite volume vacuum expectation value of the trace of the energy momentum tensor of the associated integrable quantum field theory, m the physical mass of the fundamental particle (which can be identified with the spectral gap (1)) and $S(\beta)$ its two-particle scattering amplitude depending on the rapidity β parametrizing the two-momentum $p^\mu = m(\cosh\beta, \sinh\beta)$. Formula (4) is obtained when computing the groundstate energy of the integrable quantum field theory defined through the exact S-matrix on a cylinder of circumference R. In the UV limit, $R \to 0$, the infinite bulk vacuum energy can be extracted by comparison against results from conformal perturbation theory [15, 16]. Equation (4) has to be applied with caution as it must be modified when logarithmic terms are present in the singular part of the free energy. It is, however, believed to hold true for the scaling field theories and S-matrices mentioned and the amplitude (4) has been reported in [17] based on the exact scattering matrix theory in [10]. In addition, from formula (4) it can be seen that the universal amplitude enters the vacuum expectation value of the energy momentum tensor. The latter is needed as important input information in the form factor program [18, 19]. Setting up a set of recursive functional equations it allows all multi-particle form factors of the quantum field operators to be constructed starting from their vacuum expectation values.

Our discussion of the dilute A_L models provides an alternative and independent way of calculating the universal amplitude (3) and is consistent with formula (4) involving

the scattering matrices [10] and their thermodynamic Bethe ansatz analysis. It is one of the main results of our paper.

We also reveal the close link between the exact S-matrix theory and the transfer matrix method on another level which is of practical importance in the calculation of the excitation spectrum and the spectral gaps (1). As in the bootstrap approach we proceed by constructing functional relations using entirely analytic arguments. In fact, we establish a one-to-one match between the functional relations satisfied by the excitation spectrum of the transfer matrix given in terms of elliptic functions and the scattering amplitudes given in terms of trigonometric functions. Besides crossing symmetry and unitarity, the crucial bootstrap identity characterising the S-matrices [10] is the functional relation associated with a self-fusing process $b \times b \to b$:

$$S_{bb}(\beta)S_{bb}(\beta + i\tfrac{2\pi}{3}) = S_{bb}(\beta + i\tfrac{\pi}{3}) \ . \tag{5}$$

Here S_{bb} denotes the two-particle amplitude either of the fundamental particle or of a bound state of it. This distinction is important as, except for the case $L = 3$, the fundamental particle does not necessarily display a self-fusing process. However, there are always bound states of the fundamental particle which exhibit this peculiar property [10]. The equation (5) is understood w.r.t. analytic continuation and implies a pole in the physical sheet at $\beta = i\frac{2\pi}{3}$.

In the context of the dilute A_L model we demonstrate in the thermodynamic limit that for regime 2 the bootstrap equation (5) is reflected in the excitation spectrum of the dilute A_L transfer matrix by the appearance of an analogous identity,

$$r(u)r(u + 2\lambda) = r(u + \lambda) \ , \quad r(u) = \lim_{N\to\infty} \frac{\Lambda(u)}{\Lambda_0(u)} \ . \tag{6}$$

The variable u is the spectral parameter entering the Boltzmann weights and does not have the physical interpretation of the rapidity in (5). The constant λ is the crossing parameter whose dependence on the maximal height value varies in the different branches of the model (see Table 1). We stress that the relation (6) holds for any excited state Λ not only the leading one. It has been reported before in the literature for $L = 3, 4, 6$ as an outcome of the exact perturbation theory approach [4, 20, 21, 22] based on various string hypotheses [23, 25, 24, 26].

In this paper we derive it in the massive regime for general L directly from the known eigenvalue spectrum using only analytic arguments. Functional equations of the transfer matrix in the form of an inversion relation are well-known in the literature, e.g. [27, 28, 29, 3]. However, we emphasize that the above functional identity (6) of bootstrap type is more powerful as it implies the inversion relation:

$$r(u)r(u + 3\lambda) = 1 \ . \tag{7}$$

This relation corresponds to a combination of the quite general properties of unitarity and crossing symmetry in exact S-matrix theory, while the appearance of a bootstrap equation such as (6) displays a characteristic feature of the model. For $L = 3, 4, 6, \infty$

we indeed recover all bootstrap identities by employing a generic formulation in terms of Coxeter geometry, originally applied in the context of affine Toda field theories [30, 31, 32]. The fundamental excitations of the dilute $A_{3,4,6,\infty}$ models can be cast into a single formula involving the Coxeter element of the underlying Lie algebra, just as the ADE affine Toda S-matrices for real coupling have been [32, 30]. The latter contain the aforementioned scattering matrices as special cases, with the important difference of a coupling dependent CDD factor [33, 6, 9, 34, 35].

As in the context of the bootstrap construction of scattering matrices, the functional equations describing the excitation spectrum determine it completely, up to the location of its zeroes and poles. The missing information on their position is encoded in the Bethe roots. We will argue that in the thermodynamic limit the leading excitations are related either to a hole or a two-string in the groundstate distribution of the Bethe roots. This picture is supported by revealing the underlying Coxeter geometry related to the exceptional algebras $E_{8,7,6}$ inherent to the special cases $L = 3, 4, 6$ and to the simple Lie algebra D_4 for the unrestricted model $L = \infty$ (see Table 2). The latter is related to the sine-Gordon model at special values of the coupling constant [10]. Moreover, our findings are consistent with all previous numerical investigations in the literature [23, 25, 36, 37] and the outcome of the exact perturbation theory approach for $L = 3, 4, 6$ [4, 20, 21, 22].

This article is organized as follows. In Section 2 we briefly review the definition and the finite size eigenvalue spectrum of the dilute A_L model. We also report a series expansion of the free energy density in the elliptic nome based on earlier calculations of the groundstate [3, 4]. Section 3 states our analyticity assumptions and presents the derivation of the bootstrap equation (6) determining the explicit form of the excitation spectrum in the thermodynamic limit up to its poles and zeroes. While the discussion in Section 3 applies to any excited state, Section 4 presents a concrete proposal for the Bethe root distributions of the leading excitation in regime 2 in the form of a hole and a two-string. The position of the zeroes is determined for these excitations and we calculate the spectral gaps in the eigenspectrum as well as the associated universal amplitude ratios. Section 5 compares the outcome against earlier results for $L = 3, 4, 6$ and presents new results on the unrestricted model $L = \infty$. The underlying Lie algebraic structures $E_{8,7,6}$ and D_4 are revealed by using Coxeter geometry which allows the fundamental excitations of all four models to be expressed in a single formula. We also derive numerous new identities which are in one-to-one correspondence with the bootstrap identities of affine Toda field theory. The underlying Lie algebraic structures will also be revealed in the derivation of the spectral gaps associated with all $8, 7, 6, 4$ fundamental particles. We obtain a series expansion in terms of the ordering field (the elliptic nome) yielding the corrections to the scaling behavior (1) in the off-critical regime. The powers in the expansion match the affine Lie algebra exponents of $E_{8,7,6}$ and D_4. Section 6 states our conclusions.

2 The off-critical dilute A_L model

The dilute A_L model [2] is an exactly solvable RSOS model defined on the square lattice. Each site is allowed to take one of L possible height values and the associated adjacency matrix is given by $3-A$ with A being the Cartan matrix of the Lie algebra $A_L \equiv su(L+1)$. In other words, the Boltzmann weights of a face are non-zero only if neighbouring heights coincide or differ by ± 1. Away from criticality the Boltzmann weights are parametrised in terms of Jacobi's theta functions of nome $p = e^{-\tau}$. We will frequently make use of their conjugate modulus representation,

$$\vartheta_1(u, e^{-\tau}) = \sqrt{\frac{\pi}{\tau}}\, e^{-\left(u-\frac{\pi}{2}\right)^2/\tau} E\left(e^{-2\pi u/\tau}, e^{-2\pi^2/\tau}\right) \tag{8}$$

with

$$E(z,p) = \prod_{n=1}^{\infty}(1-p^{n-1}z)(1-p^n/z)(1-p^n)\ . \tag{9}$$

The explicit expression of the Boltzmann weights will not matter in our discussion and can be found in [3]. The eigenspectrum of the row-to-row transfer matrix for a strip of length N has been reported in [23] (we choose N even),

$$\begin{aligned}\Lambda(u) \;\; = \;\; & \omega \left\{\frac{\vartheta_1(2\lambda-u,p)\vartheta_1(3\lambda-u,p)}{\vartheta_1(2\lambda,p)\vartheta_1(3\lambda,p)}\right\}^N G(u) \\ & + \left\{\frac{\vartheta_1(u,p)\vartheta_1(3\lambda-u,p)}{\vartheta_1(2\lambda,p)\vartheta_1(3\lambda,p)}\right\}^N \frac{G(u-\lambda)}{G(u-2\lambda)} \\ & + \omega^{-1}\left\{\frac{\vartheta_1(u,p)\vartheta_1(\lambda-u,p)}{\vartheta_1(2\lambda,p)\vartheta_1(3\lambda,p)}\right\}^N G(u-3\lambda)^{-1}\end{aligned} \tag{10}$$

Here we have introduced for convenience the auxiliary function

$$G(u) = \prod_{n=1}^{N} \frac{\vartheta_1(u-u_n+\lambda,p)}{\vartheta_1(u-u_n-\lambda,p)}\ . \tag{11}$$

The behaviour of the model depends crucially on the choice of parameters. Henceforth we will restrict ourselves to the regime 2 with the interval of the spectral parameter and the crossing parameter λ as specified in Table 1. The complex numbers $u_n\ ,\ n = 1, ..., N$ characterising the eigenvalue are the Bethe roots, subject to the Bethe equations

$$\mathfrak{p}(u_m) = -\omega \quad \text{with} \quad \mathfrak{p}(u) = \left\{\frac{\vartheta_1(u+\lambda,p)}{\vartheta_1(u-\lambda,p)}\right\}^N \frac{G(u)}{G(u+\lambda)G(u-\lambda)}. \tag{12}$$

where the phase factor $\omega = \exp\left(i\pi\frac{\ell}{L+1}\right)$ with $\ell = 1, \ldots, L$ depends on the particular position of the Bethe roots chosen. In fact, using the known transformation properties of the theta function

$$\vartheta_1\left(u \pm i\tau, e^{-\tau}\right) \;\; = \;\; -e^{\pm\frac{\tau}{2}}e^{\mp 2iu}\vartheta_1\left(u, e^{-\tau}\right)\ , \quad \tau > 0 \tag{13}$$

$$\vartheta_1\left(u+\pi, p\right) \;\; = \;\; -\vartheta_1\left(u, p\right) = \vartheta_1\left(-u, p\right) \tag{14}$$

one verifies that the corresponding eigenvalue $\Lambda(u)$ is invariant under the simultaneous replacement

$$u_n \to u_n \pm i\tau \quad \text{and} \quad \omega \to \omega\, e^{\mp i4\lambda} \tag{15}$$

as well as under the shift

$$u_n \to u_n \pm \pi \tag{16}$$

of a single Bethe root. Moreover, one easily deduces from the transformation properties

$$G(u) = e^{\pm 4Ni\lambda} G(u \pm i\tau) = G(u+\pi) \tag{17}$$

of the auxiliary function (11) that the eigenvalues satisfy

$$\Lambda(u+\pi) = \Lambda(u) \quad \text{and} \quad \Lambda(u \pm i\tau) = p^{\mp N} e^{\pm i6\lambda N} e^{\mp 4Niu} \Lambda(u) \tag{18}$$

while the function determining the Bethe equations (12) is doubly periodic

$$\mathfrak{p}(u) = \mathfrak{p}(u+i\tau) = \mathfrak{p}(u+\pi)\ . \tag{19}$$

Finally, from the identity $\vartheta_1\left(u, e^{i\pi}p\right) = e^{i\frac{\pi}{4}} \vartheta_1\left(u, p\right)$ it is seen that, for all finite N, the eigenspectrum and Bethe equations are invariant under the exchange $p \to e^{i\pi}p$ as only ratios of Jacobi's theta functions occur in (10) and (12). However, the Boltzmann weights [2] are in general not invariant under sign reversal of the elliptic nome, but can be related through a relabelling of the heights if L is odd. Therefore, while all eigenvalues of the transfer matrix can be parametrized through (10) and (12) with p replaced by $|p|$, an eigenvalue present for $p > 0$ might be absent for $p < 0$, and vice versa, if L is even.

2.1 The groundstate and scaling corrections

The eigenvalue Λ_0 corresponding to the groundstate of the model has been investigated for arbitrary L in [4] by applying Baxter's exact perturbation theory approach [38] in the ordered limit $|p| \to 1$. The groundstate Bethe roots are assumed to lie on the imaginary axis, which has been supported by numerical investigations. The result obtained for the groundstate is assumed to hold for any $0 < |p| < 1$ in the thermodynamic limit and allows one to compute the free energy density (2)

$$\begin{aligned} f \;\; &= \;\; -\lim_{N\to\infty} N^{-1} \ln \Lambda_0(\tfrac{3\lambda}{2}) = 4\sqrt{3} \sum_{k=0}^{\infty} \frac{\cos\left[(\frac{5\pi}{6} - \frac{\pi^2}{6\lambda})(6k+1)\right]}{(6k+1)\sin\left[\frac{\pi^2}{6\lambda}\,(6k+1)\right]} \, \frac{p^{\frac{\pi}{3\lambda}(6k+1)}}{1 - p^{\frac{\pi}{3\lambda}(6k+1)}} \\ &\quad -4\sqrt{3} \sum_{k=1}^{\infty} \frac{\cos\left[(\frac{5\pi}{6} - \frac{\pi^2}{6\lambda})(6k-1)\right]}{(6k-1)\sin\left[\frac{\pi^2}{6\lambda}\,(6k-1)\right]} \, \frac{p^{\frac{\pi}{3\lambda}(6k-1)}}{1 - p^{\frac{\pi}{3\lambda}(6k-1)}} \\ &\quad -8 \sum_{k=1}^{\infty} \frac{\sin^2 \frac{3\lambda}{2}k}{k} \, \frac{\cos 5\lambda k \cos \lambda k}{\cos 3\lambda k} \, \frac{p^{2k}}{1-p^{2k}} = f_s + f_{ns}. \end{aligned} \tag{20}$$

This formula is obtained from the result for the partition function per site derived in [3, 4] by applying the Poisson summation formula and the residue theorem. The spectral parameter has been set to the isotropic value $u = 3\lambda/2$ in order to ensure that the eigenvalues are real. We come back to this point later on. From the series expansion (20) the coefficient of the leading term in the critical limit $p \to 0^+$ is obtained as

$$\mathcal{Q}^+(0) = \lim_{p\to 0^+} p^{-\frac{1}{1-\Delta}} f_s = 4\sqrt{3}\,\frac{\cos\frac{\pi}{6\lambda}(5\lambda-\pi)}{\sin\frac{\pi}{6\lambda}\pi} \tag{21}$$

with $2\Delta = 2\Delta_{1,2} = 2 - 6\lambda/\pi$ being the scaling dimension of the perturbing operator. As already pointed out in the introduction, the latter depends on the chosen normalization of the fields and is a non-universal quantity. In order to obtain the universal quantity (3) we need to compute (1) from the leading excitation in the thermodynamic limit. To this end, in the next section we derive functional equations satisfied by any excited state in the thermodynamic limit.

3 Bootstrap identities in the dilute A_L model

Conventionally the excitation spectrum in the thermodynamic limit is calculated proceeding via the thermodynamic Bethe ansatz. Alternatively, one might proceed entirely by analytic arguments as has been demonstrated e.g. in the context of the XYZ model [29]. Rather than solving the intricate non-linear integral equations of the thermodynamic Bethe ansatz, we will instead derive a set of functional relations for the ratio

$$r(u) = \lim_{N\to\infty} \frac{\Lambda(u)}{\Lambda_0(u)}\ . \tag{22}$$

Here $\Lambda(u)$ denotes the eigenvalue of any excited state over the ground state $\Lambda_0(u)$ such that the ratio Λ/Λ_0 has a non-vanishing well-defined limit as $N \to \infty$. We will in particular show the functional equation (6), which, in conjunction with certain analyticity assumptions, imposes powerful restrictions determining the excitation spectrum up to the position of its poles and zeroes. The latter are fixed by the Bethe roots which are usually unknown. However, the functional relations we are going to discuss yield constraints on their positions.

3.1 Asymptotic behaviour in the thermodynamic limit

Recall that the Bethe roots $\{u_n^{(0)}\}$ belonging to the groundstate Λ_0 lie on the imaginary axis. It is reasonable to assume in general that for the low lying excited states Λ the associated Bethe root distribution $\{u_n\}$ is similar, and that at most a finite number of holes are present or a finite number of Bethe roots have non-vanishing real part. Well away from the critical point, $0 < |\ln p| \ll 1$, this information is sufficient to show that as $N \to \infty$ either the first, second or third term in the eigenvalue expression (10) becomes dominant as the spectral parameter sweeps through the allowed interval $0 < u < 3\lambda$.

The crucial feature exploited is the conjugate modulus transformation (8), as outlined in Appendix A. Explicitly, one finds for the groundstate as well as for the excited states the following:

$$\Lambda(u) \sim \begin{cases} \omega \left\{ \frac{\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \right\}^N G(u) \left[1+o(e^{-N})\right], & 0<u<\lambda \\ \left\{ \frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \right\}^N \frac{G(u-\lambda)}{G(u-2\lambda)} \left[1+o(e^{-N})\right], & \lambda<u<2\lambda \\ \frac{1}{\omega} \left\{ \frac{\vartheta_1(u)\vartheta_1(\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \right\}^N \frac{1}{G(u-3\lambda)} \left[1+o(e^{-N})\right], & 2\lambda<u<3\lambda \end{cases} . \tag{23}$$

By the above notation we indicate that all terms besides the leading one are exponentially suppressed in N. In order to discuss the thermodynamic limit we introduce

$$g(u) := \lim_{N\to\infty} g^N(u) , \quad g^N(u) := \frac{G(u)}{G_0(u)} , \tag{24}$$

where $G_0(u)$ denotes the auxiliary function (11) corresponding to the groundstate. By an analogous argument as in the XYZ case [29], the convergence of the above limit is expected to be uniform as only a finite number of Bethe roots have non-vanishing real part or are missing compared to the groundstate distribution. The excitation spectrum then obeys

$$r(u) = \lim_{N\to\infty} \frac{\Lambda(u)}{\Lambda_0(u)} = \begin{cases} \frac{\omega}{\omega_0}\, g(u) , & 0<\mathrm{Re}\, u<\lambda \\ g(u-\lambda)/g(u-2\lambda) , & \lambda<\mathrm{Re}\, u<2\lambda \\ \frac{\omega_0}{\omega}\, g(u-3\lambda)^{-1} , & 2\lambda<\mathrm{Re}\, u<3\lambda \end{cases} . \tag{25}$$

An obvious consequence of this limiting behaviour is the identity

$$r(u)r(u+2\lambda) = r(u+\lambda) , \quad 0<\mathrm{Re}\, u<\lambda. \tag{26}$$

In order to promote this relation to the functional equation (6) valid in the whole complex plane, we need to investigate the analytic behaviour of the excitation spectrum in the thermodynamic limit.

3.2 Analyticity and the general form of the excitation spectrum

From the explicit expression (10) for the eigenspectrum, it is inferred that, for finite N, poles and zeroes of the ratio $\Lambda(u)/\Lambda_0(u)$ are completely determined by the location of the Bethe roots. Employing the asymptotic behaviour (23) one easily deduces that for $N \gg 1$ the purely imaginary Bethe roots u_n of the excited and of the ground state give rise to poles located at

$$u^P_{\mathrm{Im}} = u_n+\lambda,\ u_n+2\lambda \bmod \pi . \tag{27}$$

These pole contributions will cancel each other when the limit $N\to\infty$ is taken, since the imaginary roots then become densely located along the imaginary axis centered at

$\lambda, 2\lambda \bmod \pi$. The only exceptions are poles in the groundstate which give rise to zeroes of the eigenvalue ratio $\Lambda(u)/\Lambda_0(u)$ when holes are present in the excited state. Bethe roots u_n of the excited state belonging to a string, i.e. $\mathrm{Re}\, u_n \neq 0$, will also give rise to isolated zeroes and poles. However, since the number of holes and strings is assumed to be finite, one infers from (10) that in each of the intervals

$$0 < \mathrm{Re}\, u < \lambda\ , \quad \lambda < \mathrm{Re}\, u < 2\lambda \quad \text{and} \quad 2\lambda < \mathrm{Re}\, u < 3\lambda \tag{28}$$

the ratio $\Lambda(u)/\Lambda_0(u)$ is analytic except for isolated points, and therefore converges towards a meromorphic function $r(u) = \lim_{N\to\infty} \Lambda(u)/\Lambda_0(u)$ in these intervals. Upon analytic continuation $r(u)$ can then be extended to the whole complex plane and the identity (26) becomes the functional equation (6) valid for all complex values of the spectral parameter. As an immediate result we have the inversion relation (7). Applying the inversion relation twice we deduce that the excitation spectrum is doubly periodic:

$$r(u) = r(u + i\tau) = r(u + 6\lambda)\ . \tag{29}$$

The first period is a direct consequence of (18) and is already present at finite N. The assumption of analyticity together with the double-periodicity property are powerful restrictions which determine the excitation spectrum up to the position of its poles and zeroes by applying the following theorem [38]:

Theorem. *Let $f(u)$ be a meromorphic function satisfying*

$$f(u + 2K) = f(u + 2iK') = f(u)$$

with K, K' being quarter periods to the elliptic modulus k. If $f(u)$ has n poles u_j^P per period rectangle then it has also n zeroes u_j^Z and can be expressed as

$$f(u) = C e^{\frac{i\pi s'}{K} u} \prod_{j=1}^{n} \frac{\vartheta_1\left[\frac{\pi}{2K}(u - u_j^Z), e^{-\pi\frac{K'}{K}}\right]}{\vartheta_1\left[\frac{\pi}{2K}(u - u_j^P), e^{-\pi\frac{K'}{K}}\right]}\ .$$

Here C is a constant and s' an integer determined by the sum rule

$$\sum_{j=1}^{n} (u_j^Z - u_j^P) = 2sK - 2is'K'\ , \quad s, s' \in \mathbb{Z}\ . \tag{30}$$

We therefore conclude that the excitation spectrum of the dilute A_L model at infinite volume is of the form

$$r(u) = C e^{\frac{i\pi s'}{3\lambda} u} \prod_{j=1}^{n} \frac{\vartheta_1\left[\frac{\pi}{6\lambda}(u - u_j^Z), |p|^{\frac{\pi}{6\lambda}}\right]}{\vartheta_1\left[\frac{\pi}{6\lambda}(u - u_j^P), |p|^{\frac{\pi}{6\lambda}}\right]}\ . \tag{31}$$

Inserting this expression in the functional relations (7) and (6) we deduce several constraints on the location of poles and zeroes. From the inversion relation it follows that

$$C^2 = 1,\ s' = 0,\ s = 3\lambda n \quad \text{and} \quad u_j^P = u_j^Z \pm 3\lambda\ . \tag{32}$$

It is therefore sufficient to locate either the zeroes or the poles. From the functional equation (6) we deduce that the existence of a zero u_j^Z implies another zero either at $u_j^Z + \lambda$ or at $u_j^Z - \lambda$. Thus, all zeroes come in pairs and n in (31) is even. Furthermore, the constant in (31) is fixed to be $C = (-1)^{n/2}$.

3.3 Hermitian analyticity and unitarity

In order to restrict the position of zeroes and poles further we derive another functional equation which is the analog of hermitian analyticity and unitarity in exact S-matrix theory [39]:

$$r(-u^*)^* = r(u)^{-1}\ . \tag{33}$$

Here $*$ denotes complex conjugation. In order to prove (33) we need to make the further assumption that the complex Bethe roots form a set which is symmetric w.r.t. to the imaginary axis modulo multiples of π, i.e. the set $\{u_n\}$ should be invariant under the mapping

$$u_n \to -u_n^* \mod \pi\ . \tag{34}$$

This is trivially fullfilled for the Bethe roots of the groundstate Λ_0, which lie on the imaginary axis. In the case of an excited state Λ a finite number of the Bethe roots $\{u_n\}$ might have non-vanishing real part and the above constraint is non-trivial. However there is numerical evidence for $L = 3, 4$ that (34) is satisfied and all string hypotheses [23, 25, 26, 22] previously reported in the literature for $L = 3, 4$ are subject to this constraint. As a consequence of (34) the auxiliary function (11) for the groundstate as well as for an excited state satisfies

$$G(-u)^* = \prod_{n=1}^{N} \frac{\vartheta_1(u^* + u_n^* - \lambda)}{\vartheta_1(u^* + u_n^* + \lambda)} = G(u^*)^{-1}\ . \tag{35}$$

From this property we infer for all N that the corresponding eigenvalues (10) are crossing symmetric:

$$\Lambda(u) = \Lambda(3\lambda - u^*)^*\ . \tag{36}$$

Upon analytic continuation in the thermodynamic limit we conclude from this crossing symmetry and the inversion relation (7) that the property (33) must hold. Thus, we can eliminate the pole variables u^P from (31) by setting $u^P = -u^{Z\,*}$. We further deduce

from crossing symmetry that the set of zeroes of the excitation spectrum (31) must be invariant under the mapping

$$u^Z \to 3\lambda - u^{Z*} . \tag{37}$$

Together with our earlier observation that the zeroes come at least in pairs $(u^Z, u^Z + \lambda)$ this now implies that one has either a pair

$$(u^Z, u^Z + \lambda) \quad \text{with} \quad \mathrm{Re}\, u^Z = \lambda \tag{38}$$

or a quartet of zeroes

$$(u^Z, u^Z + \lambda, 2\lambda - u^{Z*}, 3\lambda - u^{Z*}) . \tag{39}$$

As argued earlier, the real parts of the zeroes u^Z are determined by the complex Bethe roots u_n with $\mathrm{Re}\, u_n \neq 0$, whence the conditions (38) and (39) constitute further restrictions on the string structure of the dilute A_L model, besides (34). We leave a complete investigation of the string structure to future work [40] and now focus on the leading excitations in regime 2.

4 The leading excitations in regime 2

Having determined the general form of the excitation spectrum in regime 2 in the thermodynamic limit, we now proceed by putting forward a concrete proposal for the leading excitations which will ultimately allow us to compute the leading correlation length (1) and the universal quantity (3).

Proposal. *The leading excitation in regime 2 consists either of a hole or the presence of a two-string in the groundstate distribution of the Bethe roots.*

This proposal is supported by numerical calculations [23, 25, 36, 37], the thermodynamic Bethe ansatz [23] and the exact perturbation theory approach for the dilute A_L models in regime 2 with $L = 3, 4$ [4, 20, 21, 22]. Further support comes from extracting the correct Lie algebraic structures for $L = 3, 4, 6, \infty$, as we will discuss in the subsequent section. Here we first discuss the implications of our proposal for general L.

From (23) and (25) we infer that the zeroes of the excitation spectrum are determined by the zeroes of the auxiliary function (24) in the appropriate domain of analyticity. The latter are determined by the Bethe roots according to the definition (11). In the thermodynamic limit only the real part of the zeroes u^Z in the excitation spectrum (31) needs to be determined, since $\mathrm{Im}\, u^Z$ is linked to the imaginary part of the Bethe roots which can be parametrized by a continuous variable as $N \to \infty$. Given a Bethe root we note that $G(u)$ in general has a simple zero and pole at

$$G(u): \quad u_I^Z = u_n - \lambda \quad \text{and} \quad u_I^P = u_n + \lambda, \tag{40}$$

respectively. Accordingly, the second and third terms in the eigenvalue expression (10) exhibit zeroes and poles located at

$$\frac{G(u-\lambda)}{G(u-2\lambda)}: \quad u_{II}^Z = u_n, u_n + 3\lambda\ , \quad u_{II}^P = u_n + \lambda, u_n + 2\lambda \tag{41}$$

and

$$G(u-3\lambda)^{-1}: \quad u_{III}^Z = u_n + 4\lambda\ , \quad u_{III}^P = u_n + 2\lambda\ . \tag{42}$$

We deduce from the asymptotic behaviour (23) that in the thermodynamic limit these poles and zeroes are only of significance for the excitation spectrum if their real parts are located inside the domains of analyticity stated in (25). This constraint originates in the fact that the excitation spectrum (31) is the analytic continuation of meromorphic functions whose region of analyticity is restricted. The resulting zeroes are therefore characteristic for the regime chosen. Recall from (17) that the zeroes and poles of the auxiliary function are only fixed modulo π. The following relation for regime 2 will therefore prove important to determine the zeroes and poles of the excitation spectrum:

$$\text{regime 2}: \quad \pi = 3\lambda + \frac{\pi}{r}\ , \quad r = \Delta_{1,2}^{-1} = 4\frac{L+1}{L-2}. \tag{43}$$

Henceforth, it will always be understood that zeroes related by multiples of π can be identified.

4.0.1 Zeroes from imaginary Bethe roots and holes

Before we turn to complex Bethe roots with nonvanishing real part, we first investigate the possible zeroes and poles related to the imaginary roots in the thermodynamic limit. As we have established in (40), (41) and (42), only the poles $u_n + \lambda, u_n + 2\lambda$ may occur as $N \to \infty$. Note that these poles are located at the border of the domains of analyticity but are common to both relevant terms in (23). These poles cancel against the ones from the groundstate function unless there is a hole in the excited state in the thermodynamic limit. Then we obtain zeroes with real part located at

$$\mathrm{Re}\, u^Z = \lambda, 2\lambda \tag{44}$$

in accordance with our earlier observation (38). Whether or not a hole is present can in principle be inferred from the Bethe equations, which we will discuss below in the ordered limit. In general a hole might be accompanied by a string, i.e. some of the Bethe roots might in addition have vanishing real part. This picture has been suggested by unpublished numerical calculations [26] for $L = 4$ where the presence of a hole excitation has been reported in connection with the following 3-string:

$$v, v \pm 2\lambda\ , \quad \mathrm{Re}\, v = \tfrac{\pi}{2}. \tag{45}$$

This scenario has been investigated by the exact perturbation theory approach [22] and it was observed that the three string does not of itself contain any relevant information.

We confirm this outcome in our analytic approach; from (40), (41), and (42) we deduce for general L that the proposed 3-string does not give rise to any zeroes in the excitation spectrum.

Further restrictions on the existence of a hole excitation might originate in the maximal height value L and in the choice $p > 0$ or $p < 0$. In fact, when we evaluate these zeroes for $L = 3, 4, 6, \infty$ in a subsequent section we will see that it does not give the leading excitation for $L = 3$. This is consistent with the suggested picture from scattering theory [10] if we associate the hole-excitation with a kink state, absent for the Ising model in a magnetic field. The fundamental particle in the case when the hole excitation is not present should be associated with a particular two-string, which we consider next.

4.0.2 Zeroes induced by a two-string

A general two-string subject to the constraint (34) is of the form

$$v \pm x, \quad \mathrm{Re}\, v = 0 \text{ or } \tfrac{\pi}{2}\,, \; \mathrm{Im}\, x = 0\,. \tag{46}$$

Setting $x = \lambda/2$ and $\mathrm{Re}\, v = \pi/2$ this type of string has been observed numerically for $L = 3, 4$ [23, 25, 26]. We propose it here for general L and show below that it leads to the correct algebraic structure for the special cases $L = 3, 4, 6, \infty$. In comparison with the exact S-matrix theory proposed in [10], the associated excited state of the dilute A_L model should be identified with the lowest breather, i.e. a bound state of two kinks.

One easily verifies that $G(u)$ has simple zeroes and poles at

$$u_I^Z = v - \tfrac{3\lambda}{2}, v - \tfrac{\lambda}{2} \quad \text{and} \quad u_I^P = v + \tfrac{\lambda}{2}, v + \tfrac{3\lambda}{2}\,. \tag{47}$$

Since $\mathrm{Re}\, v = \pi/2$ the second zero has real part greater than λ and thus does not constitute a zero of the eigenvalue in the thermodynamic limit. Note also that the poles are not present in the thermodynamic limit. The second term and third terms in the eigenvalue expression (10) exhibit zeroes and poles at

$$\begin{aligned} u_{II}^Z &= v - \tfrac{\lambda}{2}, v + \tfrac{\lambda}{2}, v - \tfrac{\lambda}{2} - \tfrac{\pi}{r}, v + \tfrac{\lambda}{2} - \tfrac{\pi}{r} \\ u_{II}^P &= v + \tfrac{\lambda}{2}, \left(v + \tfrac{3\lambda}{2}\right)^2, v - \tfrac{\lambda}{2} - \tfrac{\pi}{r} \end{aligned} \tag{48}$$

and

$$u_{III}^Z = v + \tfrac{\lambda}{2} - \tfrac{\pi}{r}, v + \tfrac{3\lambda}{2} - \tfrac{\pi}{r} \quad \text{and} \quad u_{III}^P = v + \tfrac{3\lambda}{2}, v + \tfrac{5\lambda}{2}\,. \tag{49}$$

We therefore deduce the following location of the real parts of the zeroes in the excitation spectrum when the thermodynamic limit is taken:

$$\mathrm{Re}\, u^Z = \tfrac{\pi}{2r}, \lambda + \tfrac{\pi}{2r}, 2\lambda - \tfrac{\pi}{2r}, 3\lambda - \tfrac{\pi}{2r}\,. \tag{50}$$

Note that the resulting quartet fulfills the condition (39).

Having determined the zeroes of the excitation spectrum we are now in the position to write down the corresponding ratio of eigenvalues (31) and to calculate the resulting spectral gaps. Before we come to this point we further support our proposal on the excitations and their Bethe root structure by investigating the Bethe equations (12) in the ordered limit.

4.1 Bethe equations in the ordered limit

As solving the Bethe equations directly is a complicated matter, we consider them in the ordered limit $|p| \to 1$ where we can check the proposed excitations for consistency by investigating the asymptotic behavior as $N \to \infty$. We will show that the function (12) characterizing the Bethe equations is a pure phase in the massive regime for $N \gg 1$ as required. Its asymptotic behavior is determined by the real part of the Bethe roots which is the essential information entering the excitation spectrum (31). The latter depends continuously on the elliptic nome, whence we expect that our findings in the ordered limit give a qualitative picture valid for all values of $|p|$.

As we did for the eigenspectrum of the transfer matrix in Appendix A, it is also of advantage here to rewrite the Bethe equations using the conjugate modulus transformation (9):

$$\mathfrak{p}(w) = \left\{ -\frac{E(x^{2s}w)}{wE(x^{2s}/w)} \right\}^N \prod_{n=1}^{N} w_n^{\frac{2s}{r}} \frac{E(x^{2s}w/w_n)E(x^{4s}w_n/w)}{E(x^{2s}w_n/w)E(x^{4s}w/w_n)} \; . \tag{51}$$

We recall the definition of the variables, $w = e^{-\frac{2\pi}{\tau}u}$, $s = \frac{L+2}{L-2}$ and $x = e^{-\frac{\pi^2}{\tau r}}$ with $r = 4\frac{L+1}{L-2}$. The dependence on the elliptic nome x^{2r} is suppressed in the notation.

4.1.1 Bethe equations and the occurrence of holes

Let us start by investigating the ordered limit $x \to 0$ ($|p| \to 1$) when only purely imaginary Bethe roots are present. From the conjugate modulus representation (51) we infer that the leading contribution for any Bethe root $w_m = e^{-\frac{2\pi}{\tau}u_m}$ is given by

$$\mathfrak{p}(w_m) \sim (-w_m)^{-N} \prod_{n=1}^{N} w_n^{\frac{2s}{r}} = o(1) \; . \tag{52}$$

Thus, $\mathfrak{p}(w_m)$ lies on the unit circle as expected. Since we are only interested in the thermodynamic limit, the exact position of the Bethe roots does not matter. But we can extract an additional piece of qualitative information, namely that the N Bethe roots w_m are completely determined by an equation of order N. This shows that holes are absent for any finite N when only imaginary roots are present. The situation changes when we consider in addition Bethe roots which have non-vanishing real part, since they alter the asymptotic behavior in the ordered limit.

We have mentioned before the possible occurrence of the 3-string (45) for $L = 4$ [26, 22] which according to our analysis does not give rise to any zeroes in the excitation spectrum for general L. Its significance is the induction of a hole in the purely imaginary Bethe root distribution as observed in [22] for $L = 4$. We briefly review the argument for general L. Set $w_N = z\,x^{3s+1}, w_{N-1} = z'x^{-s+1}, w_{N-2} = z''x^{7s+1}$ with z, z', z'' being pure phases up to exponentially small corrections in N. As the remaining Bethe roots $w_n, n < N-2$ also lie on the unit circle, one deduces from (51) the following leading

term in the ordered limit (N even $\gg 1$),

$$\mathfrak{p}(w_m) \sim w_m^{-N+2} \frac{(zz'z'')^{\frac{2s}{r}}}{z^2} \prod_{n=1}^{N-3} w_n^{\frac{2s}{r}} = o(1) \ , \quad 1 \leq m < N-2 \tag{53}$$

As required $\mathfrak{p}(w_m)$ lies on the unit circle. In contrast to (52), however, the $N-3$ imaginary Bethe roots are now determined by an equation of order $N-2$. The extra solution points towards the existence of a hole. In addition to the Bethe equations (53) one also needs to verify that the equations for the complex Bethe roots w_N, w_{N-1}, w_{N-2} are of order one in the ordered limit. The product of the Bethe equations corresponding to the three roots in the string has to be taken in order to avoid complications in the thermodynamic limit where terms either in the numerator or denominator become small. Again one finds that the leading term lies on the unit circle in the ordered limit, $\mathfrak{p}(w_{N-2})\mathfrak{p}(w_{N-1})\mathfrak{p}(w_N) \sim o(1)$, showing that the string structure is consistent.

4.1.2 Bethe equations for the two-string excitation

We now consider the Bethe roots for the excitation associated with the two-string. We choose the first $N-2$ Bethe roots w_n to lie on the unit circle and set $w_{N-1} = z\,x^{2s+1}$, $w_N = z'x^{4s+1}$. In the thermodynamic limit we have $|z| = |z'| = 1$ and $z = z'$, thus for sufficiently large N we may assume these properties up to exponentially small corrections in N. For the Bethe roots $w_m, m < N-1$ we find $(6s+2 = 2r)$

$$\mathfrak{p}(w_m) \sim (-w_m)^{2-N} z^{\frac{4s}{r}-2} \prod_{n=1}^{N-2} w_n^{\frac{2s}{r}} = o(1) \ , \quad x \to 0, \ N \gg 1 \tag{54}$$

while the leading term for the case of the complex Bethe roots reads

$$\mathfrak{p}(w_{N-1})\mathfrak{p}(w_N) \sim (zz')^{\frac{4s}{r}-2} \prod_{n=1}^{N-2} w_n^{\frac{4s}{r}-2} = o(1) \ , \quad x \to 0, \ N \gg 1. \tag{55}$$

Thus we find for the two-string also that the arrangement of the real parts of the Bethe roots is consistent with the Bethe equations in the ordered limit. Note the absence of any holes. As indicated before, this is a preliminary check on the string structure and a more detailed analysis will be presented elsewhere [40].

4.2 Spectral gaps of the leading excitations

We now insert our results (44) and (50) on the zeroes of the excitation spectrum into the general formula (31). We recall that the imaginary part of the zeroes is undetermined, but in the thermodynamic limit becomes a continuous variable, say $\operatorname{Im} u^Z = v \in \mathbb{R}$, which parametrizes a band of eigenvalues corresponding to the excited state. In the

field theoretic context it can be identified with the rapidity. For the hole-excitation we then find

$$r_{\text{hole}}(u) = -\frac{\vartheta_1(\frac{\pi u}{6\lambda} - i\frac{\pi v}{6\lambda} - \frac{\pi}{6}, |p|^{\frac{\pi}{6\lambda}})\vartheta_1(\frac{\pi u}{6\lambda} - i\frac{\pi v}{6\lambda} - \frac{\pi}{3}, |p|^{\frac{\pi}{6\lambda}})}{\vartheta_1(\frac{\pi u}{6\lambda} - i\frac{\pi v}{6\lambda} + \frac{\pi}{6}, |p|^{\frac{\pi}{6\lambda}})\vartheta_1(\frac{\pi u}{6\lambda} - i\frac{\pi v}{6\lambda} + \frac{\pi}{3}, |p|^{\frac{\pi}{6\lambda}})} \; . \tag{56}$$

In general the eigenvalues in the excitation band will be complex and in order to determine the spectral gap, or equivalently the correlation length, one has to integrate over the whole band. However, from the crossing relation (36) we infer that the eigenvalues are real at the isotropic point $u = 3\lambda/2$. In addition, we have to tune the parameter v such that we reach the bottom of the band. The latter value is derived to be $v = \frac{\pi\tau}{12\lambda}$ and the correlation length is then obtained via the formula

$$\xi_{\text{hole}}(p)^{-1} = -\ln r_{\text{hole}}(\tfrac{3\lambda}{2})_{v=\frac{\pi\tau}{12\lambda}} = 2\ln\frac{\vartheta_4(\frac{\pi}{4} + \frac{\pi}{3}, |p|^{\frac{\pi}{6\lambda}})}{\vartheta_4(\frac{\pi}{4} - \frac{\pi}{3}, |p|^{\frac{\pi}{6\lambda}})} \; . \tag{57}$$

Using a standard identity for the elliptic functions we obtain the series expansion in the elliptic nome

$$\xi_{\text{hole}}(p)^{-1} = 4\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{6n+1}\,\frac{|p|^{\frac{\pi}{6\lambda}(6n+1)}}{1-|p|^{\frac{\pi}{3\lambda}(6n+1)}} + 4\sqrt{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{6n-1}\,\frac{|p|^{\frac{\pi}{6\lambda}(6n-1)}}{1-|p|^{\frac{\pi}{3\lambda}(6n-1)}}, \tag{58}$$

from which we easily extract the amplitude (1) of the leading term in the critical limit $p \to 0^{\pm}$,

$$\mathcal{S}^{\pm}(0)^{-1} = \lim_{p\to 0^{\pm}} |p|^{-\frac{\pi}{6\lambda}}\xi_{\text{hole}}(p)^{-1} = 4\sqrt{3} \; . \tag{59}$$

Together with the general expression for the singular part of the free energy (21) this puts us in a position to derive the universal amplitude ratio (3):

$$\mathcal{Q}^{\pm}(0)\mathcal{S}^{\pm}(0)^2 = \frac{\sin\frac{\pi}{3}\frac{L}{L+2}}{4\sqrt{3}\sin\frac{2\pi}{3}\frac{L+1}{L+2}} \; , \quad L > 3 \; . \tag{60}$$

This coincides with the result (4) from the thermodynamic Bethe ansatz analysis reported in [17] based on the exact scattering matrices constructed in [10] when the fundamental particle is related to a kink. It needs to be pointed out that the general result (60) agrees with the outcome for the special cases $L = 4, 6$ previously reported in [20]. For $L = 4$, i.e. for the leading thermal perturbation of the tricritical Ising model, (60) can also be compared against the numerical findings obtained in [41].

The case $L = 3$ is special as kinks are absent from the particle spectrum [10]. In the context of the dilute A_3 model, also, numerical investigations [23, 25] of the Bethe equations did not show evidence for the presence of a hole. As we have pointed out before, further exceptions to the hole constituting the leading excitation might occur for L even, when the sign of the elliptic nome is negated. This is, for example, the case when $L = 4$.

For these exceptional cases we therefore consider as the leading excitation the two-string originally proposed in [23] for $L = 3$. In the context of the scattering theory [10] this type of excitation would correspond to the lowest breather, which is the lightest particle exhibiting a self-fusing process. From (31) and (50) we calculate the associated spectral gap by repeating the same steps as before and find

$$\begin{aligned}
\xi_{\rm string}(p)^{-1} &= 2\ln\frac{\vartheta_4(\frac{\pi}{4}+\frac{\pi}{12}\frac{L-2}{L+2},|p|^{\frac{\pi}{6\lambda}})\vartheta_4(\frac{\pi}{4}+\frac{\pi}{12}\frac{3L+2}{L+2},|p|^{\frac{\pi}{6\lambda}})}{\vartheta_4(\frac{\pi}{4}-\frac{\pi}{12}\frac{L-2}{L+2},|p|^{\frac{\pi}{6\lambda}})\vartheta_4(\frac{\pi}{4}+\frac{\pi}{12}\frac{3L+2}{L+2},|p|^{\frac{\pi}{6\lambda}})} \\
&= 8\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\frac{|p|^{\frac{\pi}{6\lambda}n}}{1-|p|^{\frac{\pi}{3\lambda}n}}(\sin\tfrac{\pi n}{6}\tfrac{3L+2}{L+2}+\sin\tfrac{\pi n}{6}\tfrac{L-2}{L+2}) \\
&\sim 8(\sin\tfrac{\pi}{6}\tfrac{3L+2}{L+2}+\sin\tfrac{\pi}{6}\tfrac{L-2}{L+2})|p|^{\frac{\pi}{6\lambda}},\ |p|\to 0. \qquad (61)
\end{aligned}$$

We point out that the sum (61) runs at most over the integers relatively prime to 6, i.e. $n = 1, 5, 7, 11, \ldots$ These integers coincide with the allowed spins of integrals of motion compatible with a self-fusing process [7]. In order to obtain the universal amplitude (3) in the case that the hole excitation is absent we need only to determine the ratio of the two different correlation lengths in the critical limit,

$$\frac{\xi_{\rm string}(0)}{\xi_{\rm hole}(0)} = \frac{\sin\frac{\pi}{3}}{\cos\frac{2\pi}{3(L+2)}+\sin\frac{\pi}{6}\frac{L-2}{L+2}} < 1\,, \quad L > 3. \qquad (62)$$

As expected, we deduce that in presence of both a hole and a two-string the smallest spectral gap is given by the hole-excitation. In the context of the scattering matrices constructed in [10] expression (62) determines the mass ratio of the kink and the lowest breather.

5 The excitation spectrum for $L = 3, 4, 6, \infty$

For $L = 3, 4, 6$ and regime 2 the excitation spectrum has been previously put forward in the literature case-by-case, using the exact perturbation theory approach in the ordered limit [4, 20, 21, 22] and the different string hypotheses based on numerical data [23, 25, 24, 26, 36]. A significant role in these proposals is played by the underlying Lie algebraic structures related to the exceptional algebras $E_{8,7,6}$ which also arise in the associated exact scattering theories.

The excitation spectrum for the unrestricted model ($L = \infty$) connected to the algebra D_4 has not been previously considered and its formulation is a new result.

We show that our proposal of a hole-excitation and a two-string excitation leads to the correct Lie algebraic structures by connecting the location of the real part of the zeroes (44) and (50) to the Coxeter geometry of the respective algebras. Invoking similar Lie algebraic Coxeter identities as were originally applied in the context of affine Toda field theory [30, 32], we summarize the previously reported excitation spectra for $L = 3, 4, 6$ in a single formula and prove additional functional equations besides the

ones already stated for general L. These relations completely fix the location of the zeroes in (31) for the higher fundamental excitations [23, 36, 4, 20, 21, 22] as well, and can be regarded as equivalent to a string hypothesis. We start by stating the excitation spectrum in generic Lie algebraic terms for $L = 3, 4, 6, \infty$ and then identify the excitations belonging to the hole and the two-string.

5.1 Excitations in terms of Coxeter geometry

In the following let $\mathfrak{g}$ stand for one of the simple Lie algebras $E_{8,7,6}$ or D_4. Then we expect, depending on the sign of the elliptic nome, at most rank $\mathfrak{g}$ fundamental excitations or eigenvalue bands in the thermodynamic limit. Furthermore, we denote by h the Coxeter number of $\mathfrak{g}$ which obeys the following identity in terms of the maximal height value,

$$h = 6\,\frac{L+2}{L-2}\;. \tag{63}$$

Introducing the simple roots α_i and the fundamental co-weights $\lambda_i^\vee$ of the Lie algebra $\mathfrak{g}$ we define a matrix of functions $\mu : \mathbb{Z} \to \frac{1}{2}\mathbb{Z}$ by setting

$$\mu_{ij}(2a - \tfrac{c_i+c_j}{2}) = -\frac{c_i}{2}\left\langle \lambda_j^\vee, \sigma^a \alpha_i \right\rangle\;, \quad c_i = \pm 1\;. \tag{64}$$

Here σ denotes the Coxeter element of the Weyl group of $\mathfrak{g}$. It is uniquely defined w.r.t. to a specific bicolouration of the Dynkin diagram specified by the "colours" c_i. See Appendix B for details. We are now prepared to state the excitation spectrum for $L = 3, 4, 6, \infty$:

$$r_j(u) = (-1)^{A^{-1}_{\star j}} \prod_{a=1}^{h} \left\{ \frac{\vartheta_1\left(\frac{h+2}{2h}u - \frac{\pi a}{2h} - i\frac{\pi\alpha}{2h}, |p|^{\frac{h+2}{2h}}\right)}{\vartheta_1\left(\frac{h+2}{2h}u + \frac{\pi a}{2h} - i\frac{\pi\alpha}{2h}, |p|^{\frac{h+2}{2h}}\right)} \right\}^{2\mu_{\star j}(a)}\;. \tag{65}$$

Here $j = 1, 2, \ldots$rank $\mathfrak{g}$ runs over the fundamental particle excitations and the index $\star$ specifies the root in the Dynkin diagram of the simple Lie algebra $\mathfrak{g}$ which is connected to the affine root in the Dynkin diagram of the affine extension $\mathfrak{g}^{(1)}$ [21]. We will show below that this particular node is singled out by the two-string. The variable $\alpha \in \mathbb{R}$ parametrizes the imaginary part of the Bethe roots in the thermodynamic limit and scales the eigenvalue bands. The sign factor in front is determined by the inverse Cartan matrix which is given in terms of the simple roots as $A_{kl} = 2\langle\alpha_k, \alpha_l\rangle / \langle\alpha_k, \alpha_k\rangle$. In fact, the Cartan matrix allows for an alternative definition of the exponent function (64), which will be of great practical use in the subsequent computations. Consider the q-deformed Cartan matrix $[A]_q$ defined by

$$([A]_q)_{kl} := [A_{kl}]_q\;, \quad [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{66}$$

Then the functions (64) can be implicitly defined by the relation [42, 43]

$$\frac{1-q^{2h}}{2}[A]_q^{-1} = \sum_{a=1}^{2h} \mu(a) q^a \ . \tag{67}$$

The above relation is non-trivial as it states that the inverse q-deformed Cartan matrix, which generically is a rational function in q, simplifies to a polynomial when multiplied with the factor $(1-q^{2h})$. The functions μ then simply count how often the monomial q^a occurs in this expansion. That the two definitions of μ are indeed equivalent has been proven in [42, 43]. One can also now check explicitly that the above expression (65) correctly reproduces the case-by-case results derived previously [4, 20, 21, 22]*. There and in [36] it was noticed that the characteristic integers a fixing the real part of the zeroes in the excitation spectrum (65) match the integers appearing in the building blocks of hyperbolic functions used in describing the ADE affine Toda S-matrices [9]. In the context of affine Toda field theory the relation of these integers to Coxeter geometry was established in [30, 32].

One can now explicitly check that the resulting zeroes (50) and (44) from the hole and two-string excitation match the Lie algebraic formula (65) when the node $\star$ is specified by the simple root connected to the affine root of $\mathfrak{g}$. Employing the helpful identities

$$\lambda = \frac{\pi}{2r}\frac{h}{3} \quad \text{and} \quad 2r = 2\Delta_{1,2}^{-1} = h+2$$

one finds the real parts of the zeroes listed in Table 3 (a) and 3 (b). We note that the hole gives always the lightest particle in the mass spectrum (except for $L=3$) while the two-string associated with the node $\star$ corresponds to the lightest particle which exhibits a self-fusing process (5). It is remarkable that from the simple excitation of a hole and a two-string one already sees the four different Lie algebraic structures emerging.

L	algebra	$\frac{h+2}{\pi}\,\mathrm{Re}\,u^Z$
4	E_7	$6, 12$
6	E_6	$4, 8$
∞	D_4	$2, 4$

(a)

L	algebra	$\star$	$\frac{h+2}{\pi}\,\mathrm{Re}\,u^Z$
3	E_8	1	$1, 11, 19, 29$
4	E_7	2	$1, 7, 11, 17$
6	E_6	2	$1, 5, 7, 11$
∞	D_4	2	$1, 3^2, 5$

(b)

Table 3 (a) and 3 (b). Real parts of the zeroes in units $\frac{\pi}{h+2}$ describing the excitation spectrum corresponding to the presence of a hole (a) and the two-string (b). The results are in accordance with the general formula (65) and display the Coxeter geometry of the underlying Lie algebra. Note that the hole excitation corresponding to the kink is absent for L=3. For L>3 this excitation corresponds to the particle index j=1. The numeration of the nodes is in accordance with the conventions in [4, 20, 21, 22].

*Setting $\alpha = \frac{h+2}{2\pi}\tau$ we obtain the leading contribution previously reported in the literature.

The excitations (65) with particle indices $j > 2$ ought to belong to heavier particles which can be expressed as bound states of the former ones. This suggest that the corresponding Bethe roots distributions, which might contain additional strings or holes, have to fulfill further constraints originating from bootstrap identities other than (6). At the moment we leave further exploration of the string structure of the dilute A_L models to future work [40] and simply state these conjectured excitations as additional consistent solutions to the equation (6) which naturally arise from the Lie algebraic structure. Nevertheless, as a preparatory step for a closer investigation of the string structure, we state the additional bootstrap identities arising from Coxeter geometry in the next section.

5.1.1 Bootstrap identities

We start by verifying that the explicit expression (65) for $L = 3, 4, 6, \infty$ obeys the functional relations (33), (36) and (6). To this end it is helpful to introduce the meromorphic function

$$t(u,a) := \frac{\vartheta_1\left(\frac{h+2}{2h}u - \frac{\pi a}{2h}, |p|^{\frac{h+2}{2h}}\right)}{\vartheta_1\left(\frac{h+2}{2h}u + \frac{\pi a}{2h}, |p|^{\frac{h+2}{2h}}\right)} \tag{68}$$

which enjoys the following transformation properties

$$\begin{aligned} t(u+6\lambda,a) &= t(u,a\pm 2h) = t(-u,a)^{-1} = t(u,-a)^{-1} = t(u,a) && (69)\\ t(u+3\lambda,a) &= -t(u,h-a)^{-1} && (70)\\ t(u+i\tau,a) &= e^{2\pi i\frac{a}{h}}t(u,a)\ . && (71) \end{aligned}$$

Using the above identities the verification of the functional equations (33), (36) and (6) becomes a purely algebraic problem which can be formulated entirely in terms of the generating function (64). The latter has been thoroughly investigated in the context of affine Toda field theory [32, 42, 43] and proceeding in an analogous manner one can establish a one-to-one match between the functional relations satisfied by the excitation spectrum of the dilute $A_{3,4,6,\infty}$ model and the functional identities obeyed by the factorizable scattering matrices of the corresponding affine Toda field theory.

The properties of the generating function (64) which we exploit are [32, 42, 43]

$$\mu(a) = \mu(2h+a) = -\mu(2h-a)\ . \tag{72}$$

An additional remarkable identity which is not straightforward to prove reads [32, 42, 43]

$$\mu_{kl}(h-a) = \mu_{\bar{k}l}(a), \tag{73}$$

where $\bar{k}$ denotes the node in the Dynkin diagram of the Lie algebra $\mathfrak{g}$ which is obtained by applying a possible Dynkin diagram automorphism to the node k. The second definition of the generating function (67) is most useful for verifying the formulas

$$\mu(a) = \mu(a)^t \quad \text{and} \quad \mu(a+1) + \mu(a-1) = (2-A)\mu(a)\ . \tag{74}$$

We note that the relation (33) corresponding to hermitian analyticity in exact S-matrix theory is trivially satisfied due to (69). The inversion (7) and crossing relation (36) then follow by observing that the preferred node $\star$ is self-conjugate, i.e. $\star = \bar{\star}$, and employing (70). The number of zeroes is given by the inverse Cartan matrix element $A^{-1}_{k\star}$, which in the case of the E-type algebras coincides with the Kac or Dynkin labels:

$$A^{-1}_{k\star} = \sum_{a=1}^{h} 2\mu_{\star k}(h-a)\frac{a}{h} = \sum_{a=1}^{h} 2\mu_{\star k}(a)\frac{a}{h}\ . \tag{75}$$

From this algebraic identity together with (71) we now also read off the double-periodicity (29) of the excitation spectrum (65).

The central equation (6) requires a deeper algebraic analysis using Coxeter geometry. In fact, it is a special case of more general identities, which, along the lines of the analogous analysis in affine Toda field theory [30, 32], can be formulated as follows (see Appendix B on Coxeter geometry for the explanation of the various symbols):

Whenever there exist three elements in the Coxeter orbits Ω_l, associated with the simple roots $\gamma_l = c_l\alpha_l,\ c_l = \pm 1$ and $l = i, j, k$ which sum to zero, i.e.

$$\sum_{l=i,j,k} \sigma^{\xi_l}\gamma_l = 0 \quad \Leftrightarrow \quad \sum_{l=i,j,k} \mu(a \pm \eta_l) = 0\ , \quad \eta_l := -2\xi_l$$

for some triplet of integers ξ_l, one has the identity

$$\prod_{l=i,j,k} r_l(u + 3\lambda\tfrac{\eta_l}{h}) = 1. \tag{76}$$

For $i = j = k$ we recover equation (6) setting $\eta_l = 0, 2h/3, 4h/3$. In the context of affine Toda field theory these "bootstrap" identities occur when an intermediate bound state is formed, $i + j \to \bar{k}$ with $\bar{k}$ denoting the antiparticle of k. We see that in general these identities involve different particle indices. A few examples are listed in Table 4.

L	Coxeter identity	bootstrap equation
3	$\alpha_2 + \sigma^{13}\alpha_3 + \sigma^{23}\alpha_1 = 0$	$r_2(u + \frac{23}{5}\lambda)r_3(u + 2\lambda)r_1(u) = 1$
4	$\alpha_5 + \sigma^{7}\alpha_7 + \sigma^{14}\alpha_4 = 0$	$r_5(u + \frac{14}{3}\lambda)r_7(u + \frac{7}{3}\lambda)r_4(u) = 1$
6	$\alpha_1 + \sigma^{5}\alpha_3 + \sigma^{10}\alpha_2 = 0$	$r_1(u + 5\lambda)r_3(u + \frac{5}{2}\lambda)r_2(u) = 1$

Table 4. Displayed are three bootstrap identities for the excitations of the dilute $A_{3,4,6}$ models and the corresponding Coxeter identities in the root systems of the associated $E_{8,7,6}$ algebras. The numeration of the nodes is in accordance with the convention in [4, 20, 21, 22].

In the present context, these bootstrap identities completely fix the position of the poles and zeroes and are therefore equivalent to a string hypothesis. Only the identity (6) is common to all models; the remaining bootstrap equations characterize the dilute A_L model for different $L = 3, 4, 6, \infty$. They have a purely geometrical origin in the

structure of the underlying algebra's root system and we conclude that just as the bound state structure of affine Toda field theory can be encoded in Coxeter geometry, so can the string structure of the dilute $A_{3,4,6,\infty}$ models. The underlying Lie algebra also becomes manifest in another identity [42, 43] which can be built up using the more "fundamental" bootstrap identities (76):

$$r_j(u+\tfrac{\pi}{h+2})r_j(u-\tfrac{\pi}{h+2})=\prod_{l=1}^{\operatorname{rank}\mathfrak{g}} r_l(u)^{(2-A)_{jl}}\ . \tag{77}$$

Here A is the Cartan matrix of the particular Lie algebra in question. This identity directly reflects the Dynkin diagram structure. Like the more general bootstrap identities, (76) it points towards a relation between strings belonging to different excitations which we will investigate in a forthcoming publication [40].

5.1.2 Spectral gaps and affine Toda mass spectra

The excitation spectrum (65) explicitly states that in regime 2 there are at most 8,7,6 or 4 continuous bands of eigenvalues of the transfer matrix for $L=3,4,6,\infty$, respectively. Each band belonging to one of the leading excitations is separated from the groundstate by a gap. We now calculate the associated critical amplitudes and corrections to the scaling behavior (1) for all conjectured fundamental excitations (65). It will be worthwhile to do this in the framework of Coxeter geometry in order to reveal the Lie algebraic structure in the scaling corrections and relate them to the affine Toda mass spectra.

As before we evaluate the excitations at the isotropic point $u=3\lambda/2$ where the eigenvalues are real and set the parameter α in (65) to the value $\alpha_0=(h+2)\tau/\pi$ giving the lowest eigenvalue in each band. We then find

$$\xi_k(p)^{-1}=-\ln r_k(\tfrac{3\lambda}{2})_{\alpha=\frac{h+2}{\pi}\tau}=\sum_{a=1}^{h}2\mu_{\star k}(a)\ln\frac{\vartheta_4\left(\frac{\pi}{4}+\frac{\pi a}{2h},|p|^{\frac{h+2}{2h}}\right)}{\vartheta_4\left(\frac{\pi}{4}-\frac{\pi a}{2h},|p|^{\frac{h+2}{2h}}\right)}\ . \tag{78}$$

Since we are interested in the behaviour of the correlation length in the critical limit, it is favorable to rewrite the above expression in terms of a power series in the elliptic nome using a standard identity for the logarithm of Jacobi's theta functions,

$$\xi_k(p)^{-1}=4\sum_{n=1}^{\infty}\frac{\sin\frac{\pi n}{2}}{n}\frac{|p|^{\frac{h+2}{2h}n}}{1-|p|^{\frac{h+2}{h}n}}\sum_{a=1}^{h}2\mu_{\star k}(a)\sin\frac{\pi a}{h}n\ . \tag{79}$$

From the coefficients in this expansion, which contain the Coxeter integers, we need now to extract the affine Toda masses. This has been done in the literature [20] for $L=3,4,6$ case-by-case using trigonometric identities. It is worthwhile to review these calculations using Coxeter geometry to reveal the underlying Lie algebraic structure. To

this end we note the following formula for the residue of the inverse q-deformed Cartan matrix:

$$\operatorname{Res}\,[A]^{-1}_{q=e^x}\Big|_{\frac{i\pi s}{h}} = \frac{(-1)^s i}{h}\sum_{a=1}^{h} 2\mu(a)\sin\frac{\pi a}{h}s = -\frac{iP^s}{2\sin\frac{\pi s}{h}}\;. \tag{80}$$

Here $1 \leq s \leq h-1$ is an exponent of the Lie algebra in question and we have used the identity (67). The second equality in (80) involves the matrix

$$P^s = \frac{1}{2\pi i}\oint_{|\zeta-2\cos\frac{\pi s}{h}|=\varepsilon} d\zeta\;(\zeta-2+A)^{-1} \tag{81}$$

which is the orthogonal projector onto the eigenspaces of the (non-deformed) Cartan matrix,

$$(P^s)^2 = P^s \quad\text{and}\quad AP^s = 4\sin^2\frac{\pi s}{2h}\,P^s\;. \tag{82}$$

The matrix elements of the orthogonal projector consist of the components of the normalized eigenvectors. Setting $s=1$ we obtain the components of the Perron-Frobenius eigenvector which yields the (classical) affine Toda masses [31]

$$P^{s=1}_{kl} = m_k m_l\;. \tag{83}$$

Since all eigenvalues of the Cartan matrix are given in terms of the exponents, we conclude that the residue (80) vanishes if s is an integer which does not belong to the set of exponents. The final result for the correlation length and the leading term in the critical limit $p \longrightarrow 0$ therefore reads

$$\xi_k(p)^{-1} = \sum_{\hat{s}} \frac{2h}{\hat{s}}\frac{\sin\frac{\pi\hat{s}}{2}}{\sin\frac{\pi\hat{s}}{h}}\,P^{\hat{s}}_{\star k}\,\frac{|p|^{\frac{h+2}{2h}\hat{s}}}{1-|p|^{\frac{h+2}{h}\hat{s}}} \sim 2h\,\frac{m_\star m_k}{\sin\frac{\pi}{h}}\,|p|^{\frac{h+2}{2h}} + \ldots \tag{84}$$

Here $\hat{s}$ runs over the affine exponents, i.e. the finite exponents s modulo multiples of the Coxeter number, $\hat{s} = s \bmod h$. They are given in Table 5.

Algebra	exponents s
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_6	$1, 4, 5, 7, 8, 11$
D_4	$1, 5, 7, 11$

Table 5. The exponents of particular finite Lie algebras, which determine the eigenvalues of the corresponding Cartan matrix.

Note that only the odd exponents contribute to the sum in (84) and that we have chosen a normalization for the lightest mass different from $m_1 = 1$. Together with the result for the free energy density (21) the universal quantity (3) is expressed as

$$\mathcal{Q}^{\pm}(0)\mathcal{S}^{\pm}(0)^2 = 4\sqrt{3}\frac{\cos\pi\frac{h-3}{3h}}{\cos\frac{\pi}{h}}\left(\frac{\sin\frac{\pi}{h}}{2h\,m_\star m_1}\right)^2 . \tag{85}$$

The higher powers of the elliptic nome in the scaling functions (84) originate from irrelevant operators of the minimal CFT $M_{L,L+1}$ and the specific choice of the scaling variables. For the Ising model in a magnetic field, both contributions to the renormalization behavior have been analyzed in [44]. We therefore briefly comment on the scaling corrections in this case.

The scaling functions for $L = 3$. Setting $L = 3$ in (84) we explicitly write out the first scaling corrections to the spectral gap of the dilute A_3 model [45],

$$|p|^{-\frac{8}{15}}\xi_k^{-1} = \mathcal{S}^{\pm}(0)^{-1}\left(1 + |p|^{\frac{16}{15}} + |p|^{\frac{32}{15}} + S_1|p|^{\frac{48}{15}} + ...\right) . \tag{86}$$

In comparison with [44] we note the absence of the analytic terms. This absence might be explained by a different choice of scaling variables, i.e. the identification of the elliptic nome with the magnetic field in the Ising model differs from [44]. One has to keep in mind that the dilute A_L model is not necessarily equivalent to the Ising model at the critical point but only resides in the same universality class. In contrast, the terms depending on the irrelevant operators of the identity family in $M_{3,4}$ are present [44]. The latter are related to the components $T\bar{T}, T^2, \bar{T}^2$ of the energy momentum tensor.

From the expansion (20) we may also compare the scaling corrections for the singular part of the free energy density with those of [44],

$$|p|^{-\frac{16}{15}}f_s = \mathcal{Q}^{\pm}(0)\left(1 + Q_1|p|^{\frac{16}{15}} + Q_2|p|^{\frac{32}{15}} + Q_3|p|^{\frac{48}{15}} + ...\right)$$

Once more we find all terms originating from the irrelevant operators belonging to the class of the identity field reported in [44].

6 Conclusions

In this article we have put forward the leading excitations of the dilute A_L model in regime 2 as consisting of a hole and a two-string. Our proposal is for special values of L supported by numerical computations and further support comes from revealing the expected Lie algebraic structures $E_{8,7,6}$ and D_4 for $L = 3, 4, 6$ and $L = \infty$, i.e. the unrestricted model. A crucial role in exhibiting these structures is played by analyticity arguments and Coxeter geometry which have allowed us to relate the zeroes in the excitation spectrum induced by the Bethe roots directly to the underlying Lie algebra and to cast the excitation spectrum of all four cases into a single formula (65). This procedure

is more direct than using the non-linear integral equations of the thermodynamic Bethe ansatz, where one relates the corresponding integral kernels to the q-deformed Cartan matrix in Fourier space as has been done for $L = 3$ [23]. We note that the excitation spectrum (65) for E_8 in fact matches the thermodynamic Bethe ansatz outcome of [23] and the result of the exact perturbation theory approach [4] but we leave a closer investigation of the proposed string structure for the higher excitations to future work [40]. We believe that the one-to-one correspondence between the functional equations (76) and the bootstrap identities of the respective scattering matrices will be of central significance in gaining a thorough understanding of the allowed strings and a possible classification of the excitations.

Our result for the spectral gaps and the critical amplitudes connected with the correlation length can now be compared against the result (4) from the thermodynamic Bethe ansatz analysis of the relativistic scattering matrices. We briefly report the result for the cases $L = 3, 4, 6$ where the associated scattering matrices can be written in the following universal formula (see e.g. [42, 43] for a derivation):

$$S(\beta) = \exp \int_0^\infty \frac{dt}{t} \, 4 \cosh t \, \left([A]^{-1}_{q=e^t} \right)_{11} \sinh \frac{h\beta}{i\pi} t \, . \tag{87}$$

We note that the q-deformed Cartan matrix $[A]_q$ which we frequently used in our analysis of the transfer matrix excitation spectrum also appears here. A similar calculation to that we have performed in section 4 in the context of elliptic functions, yields upon invoking formula (4)

$$\mathcal{Q}^\pm(0)\mathcal{S}^\pm(0)^2 = \frac{\tan \frac{\pi}{h}}{4h \, m_1^2} \, , \tag{88}$$

which is found to coincide with our result (85). Here m_1 denotes the smallest component of the Perron-Frobenius eigenvector of the Cartan matrix, which can be identified with the classical mass. For general height values L our result (60) confirms the result in [17] based on the thermodynamic Bethe ansatz analysis of Smirnov's scattering matrices [10] when the hole excitation is identified with the presence of a kink state. As the universal amplitude enters directly into the bulk vacuum expectation value of the energy-momentum tensor needed in the form factor approach to construct correlation functions, this underlines the intimate relationship between integrable lattice models and field theories.

Another field theoretic aspect which needs to be mentioned is the interpretation of the particle spectra in light of the conformal structure at the critical point. For the special cases $L = 3, 4, 6$ where an underlying exceptional Lie algebra is present, the particle spectra manifest themselves in Rogers-Ramanujan identities for the conformal characters [46, 47, 48, 36]. Whether similar identities can be found for general L and further information on the conformal field theory can be extracted from the excitation spectrum are open challenges.

We close by pointing out that further universal amplitude ratios can be derived from the critical amplitudes we have computed in this article. For example, provided that L

is even the elliptic nome can be identified with the reduced temperature up to a metric factor, $p \equiv -t = 1 - T/T_c$. This allows one to compute the following universal quantity involving the (singular) specific heat C_s [1],

$$R_\xi^\pm = \mathcal{A}^{\frac{1}{2}} \mathcal{S}^\pm(0) \ , \quad C_s = -\frac{\partial^2 f_s}{\partial t^2} \sim \frac{\mathcal{A}}{\alpha} t^{-\alpha} \ . \tag{89}$$

Employing the concept of hyperscaling, the critical exponent α controlling the divergence of the specific heat near the critical point can be related to the critical exponent controlling the divergence of the correlation length:

$$2\nu = 2 - \alpha = (1 - \Delta_{1,2})^{-1} = \frac{4}{3} \frac{L+1}{L+2} \ , \quad L \text{ even} \ . \tag{90}$$

The coefficient $\mathcal{A}$ in (89) is easily obtained from the expansion of the singular free energy density (21) to be

$$\mathcal{A} = 4\sqrt{3}\, \alpha(1-\alpha)(2-\alpha) \frac{\sin \frac{\pi}{3} \frac{L}{L+2}}{\sin \frac{2\pi}{3} \frac{L+1}{L+2}} \ . \tag{91}$$

The important observation in this context is that for L even the critical amplitudes $\mathcal{S}^\pm(0)$ of the correlation length above ($p < 0$) and below ($p > 0$) the critical temperature do not necessarily coincide. This is indeed the case for $L = 4$, i.e. the leading thermal perturbation of the tricritical Ising model, where the particle corresponding to the hole excitation is absent for $p > 0$ and the heavier particle associated with the two-string gives the leading contribution [22]. The different universal amplitudes above and below the critical temperature are related through the ratio (62) which for $L = 4$ coincides with the previously reported result [20, 22, 41]

$$\mathcal{S}^+(0)/\mathcal{S}^-(0) = 1/2 \cos \tfrac{5\pi}{18} \ . \tag{92}$$

The absence of the hole excitation in depending on the sign of the elliptic nome should follow from symmetry arguments similar to those in the field theoretic context [49, 35]. To see this directly from the Bethe ansatz is an interesting problem for future investigations and requires a deeper analysis of the Bethe equations of the dilute A_L model in the thermodynamic limit and of the string structure of the excited states. Finally, we expect that the analysis of the excitation spectrum via functional equations as outlined in this work can also be applied to the other regimes of the dilute A_L model, though in a possibly modified form. The relevant universal amplitude ratios (3) and (89) for regime 1, corresponding to the $\phi_{2,1}$ perturbation, have recently been reported in [50].

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A The eigenspectrum after the conjugate modulus transformation

We briefly sketch how to prove the asymptotic behavior (23) in the ordered limit $|p| \to 1$. Exploiting the conjugate modulus representation (9) of Jacobi's theta functions the eigenvalue spectrum (10) can be rewritten as

$$
\begin{aligned}
(-e^{2u^2/\tau} w^{\frac{3s}{r}})^N \Lambda(w) = \omega \left[\frac{E(x^{4s}/w)E(x^{6s}/w)}{E(x^{4s})E(x^{6s})} \right]^N \prod_{j=1}^{N} w_j^{1-\frac{2s}{r}} \frac{E(x^{2s}w/w_j)}{E(x^{2s}w_j/w)} \\
+ \left[\frac{x^{2s}E(w)E(x^{6s}/w)}{wE(x^{4s})E(x^{6s})} \right]^N \prod_{j=1}^{N} \frac{w_j E(w/w_j)E(x^{6s}w_j/w)}{E(x^{2s}w_j/w)E(x^{4s}w_j/w)} \\
+ \omega^{-1} \left[\frac{wE(w)E(x^{2s}/w)}{x^{6s-2r}E(x^{4s})E(x^{6s})} \right]^N \prod_{j=1}^{N} w_j^{\frac{2s}{r}-1} \frac{E(x^{8s-2r}w_j/w)}{E(x^{4s}w_j/w)} \; .
\end{aligned}
$$

Here we have introduced the variables $w = e^{-\frac{2\pi}{\tau}u}$, $s = \frac{L+2}{L-2}$ and $x = e^{-\frac{\pi^2}{\tau r}}$ with $r = 4\frac{L+1}{L-2}$ in regime 2 with $p > 0$. The nome $q^2 = x^{2r}$ is suppressed in the notation. (Note that the parameters r and s and hence the variable x have been rescaled from those used by the authors of [37] in their various papers.) Assuming that for the groundstate all Bethe roots lie on the unit circle we derive for $0 < \tau \ll 1$ ($x \ll 1$),

$$
\frac{e^{\frac{2u^2 N}{\tau}} \Lambda_0(w)}{(-w^{-\frac{3s}{r}})^N} \sim
\begin{cases}
\omega_0 \prod_n w_n^{1-\frac{2s}{r}} + (\frac{x^{2s}}{w})^N \prod_n w_n + \omega_0^{-1} x^{2sN} \prod_n w_n^{\frac{2s}{r}}, & \frac{\pi}{L+1} < u < \lambda \\
\omega_0 (\frac{w}{x^{2s}})^N \prod_n w_n^{-\frac{2s}{r}} + 1 + \omega_0^{-1} (\frac{x^{4s}}{w})^N \prod_n w_n^{\frac{2s}{r}}, & \lambda < u < 2\lambda \\
\omega_0 x^{2sN} \prod_n w_n^{-\frac{2s}{r}} + (\frac{w}{x^{4s}})^N \prod_n w_n + \omega_0^{-1} \prod_n w_n^{\frac{2s}{r}-1}, & 2\lambda < u < 3\lambda
\end{cases}
$$

and

$$
\frac{e^{\frac{2u^2 N}{\tau}} \Lambda_0(w)}{(-w^{-\frac{3s}{r}})^N} \sim \omega_0 \prod_n w_n^{1-\frac{2s}{r}} + (\frac{x^{2s}}{w})^N \prod_n w_n + \omega_0^{-1} (\frac{w}{x^{6s-2r}})^N \prod_n w_n^{\frac{2s}{r}-1}, \quad 0 < u < \frac{\pi}{L+1} \; .
$$

We therefore have the following asymptotic behaviour as $N \to \infty$:

$$
\frac{e^{\frac{2u^2 N}{\tau}} \Lambda_0(w)}{(-w^{-\frac{3s}{r}})^N} \sim
\begin{cases}
\omega_0 \prod_n w_n^{1-\frac{2s}{r}}, & 0 < \mathrm{Re}\, u < \lambda \\
1, & \lambda < \mathrm{Re}\, u < 2\lambda \\
\omega_0^{-1} \prod_n w_n^{\frac{2s}{r}-1}, & 2\lambda < \mathrm{Re}\, u < 3\lambda
\end{cases} \; .
$$

We thus conclude that in the thermodynamic limit $N \gg 1$,

$$\Lambda_0(u) \sim \begin{cases} \omega_0 \left\{ \frac{\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \right\}^N G_0(u) \left[1 + o(e^{-N})\right], & 0 < u < \lambda \\ \left\{ \frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \right\}^N \frac{G_0(u-\lambda)}{G_0(u-2\lambda)} \left[1 + o(e^{-N})\right], & \lambda < u < 2\lambda \\ \omega_0^{-1} \left\{ \frac{\vartheta_1(u)\vartheta_1(\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \right\}^N G_0(u-3\lambda)^{-1} \left[1 + o(e^{-N})\right], & 2\lambda < u < 3\lambda \end{cases} .$$

Since the excited states $\Lambda(u)$ differ only by a finite number of Bethe roots with non-vanishing real part we obtain analogous asymptotic behaviour in the thermodynamic limit for them also. This proves (23) and therefore the functional equation (26).

B Coxeter geometry

Let $\mathfrak{g}$ denote in the following a simple Lie algebra and denote by $\{\alpha_1, ..., \alpha_n\}$ a set of simple roots which span its root system, i.e. the eigenvalues of the Cartan subalgebra in the adjoint representation. Each simple root α_i can be interpreted as element in an Euclidean vector space $\mathbb{R}^n$ of dimension $n = \text{rank}\,\mathfrak{g}$. It naturally defines a reflection $\mathbb{R}^n \to \mathbb{R}^n$ at its associated hyperplane by setting

$$v \to \sigma_i v := v - \frac{2\langle v, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i , \quad v \in \mathbb{R}^n. \tag{93}$$

The transformations σ_i are called simple Weyl reflections and generate the Weyl group W. Their action on simple roots is described in terms of the Cartan matrix associated with $\mathfrak{g}$,

$$\alpha_j \to \sigma_j \alpha_i := \alpha_i - A_{ij}\alpha_j , \quad A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} .$$

There exists a longest element in W in the sense that it is built up from a maximal number of simple Weyl reflections. It is called the Coxeter element or transformation and is defined by the product over all simple Weyl reflections, $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$. Clearly this definition depends on the ordering of the reflections, which is a matter of choice. The Coxeter element is therefore only defined up to conjugacy within the Weyl group. Following [31] a unique Coxeter element can be determined by introducing the concept of bicolouration for Dynkin diagrams. To every vertex in the Dynkin diagram we assign a colour $c_j = \pm 1$ such that two vertices linked to each other are differently coloured. This bicolouration polarizes the index set $\Delta = \{1, ..., n\}$ into two subsets $\Delta_\pm$ and allows unambiguous specification of the following Coxeter element:

$$\sigma := \sigma_-\sigma_+, \qquad \sigma_\pm := \prod_{i \in \Delta_\pm} \sigma_i . \tag{94}$$

Any Coxeter element shares the following properties [51]:

(C1) *The Coxeter element fixes no non-zero vector.*

(C2) *It is of finite order, $\sigma^h = 1$, where h is the Coxeter number defined as the sum over the Kac or Dynkin labels,*

$$h := 1 + \sum_{i=1}^{\mathrm{rank}\,\mathfrak{g}} n_i \ . \tag{95}$$

Thus, the Coxeter element permutes the roots in orbits of length h.

(C3) *The eigenvalues of σ are of the form*

$$\exp \frac{i\pi s_j}{h} \ , \quad j = 1, ..., \mathrm{rank}\,\mathfrak{g}$$

where the characteristic set of integers $1 = s_1 \leq s_2 \leq ... \leq s_n = h - 1$ are called the exponents of the Lie algebra $\mathfrak{g}$ satisfying the relation $s_{n+1-i} = h - s_i$.

The action of the Coxeter element on the root system splits the latter into orbits. A convenient choice of representatives of these orbits are the ,,coloured" simple roots, $\gamma_i = c_i \alpha_i$. The Coxeter orbits Ω_i defined as

$$\Omega_i := \{\sigma^x \gamma_i : 1 \leq x \leq h\} \tag{96}$$

satisfy the crucial property that they do not intersect, i.e. $\Omega_i \cap \Omega_j = \emptyset$, and are exhaustive on the set of roots. Moreover, all γ_i's lie in different orbits and all elements in one orbit are linearly independent [31]. Thus, the coloured simple roots constitute a complete set of representatives for the Coxeter orbits Ω_i.

References

[1] V. Privman, P.C. Hohenberg, A. Aharony, *Universal Critical-Point Amplitude Relations*, in Phase Transitions and Critical Phenomena (vol. 14), eds. C. Domb and J.L. Lebowitz, Academic Press, London (1991).

[2] S. O. Warnaar, B. Nienhuis and K. A. Seaton, *Phys. Rev. Lett.* **69** (1992) 710.

[3] S. O. Warnaar, P. A. Pearce, K. A. Seaton and B. Nienhuis, *J. Stat. Phys.* **74** (1994) 469.

[4] M. T. Batchelor and K. A. Seaton, *Nucl. Phys.* **B520** (1998) 697.

[5] A. B. Zamolodchikov, *Int. J. Mod. Phys.* **A 4**, 4235 (1989); Adv. Stud. Pure Math. **19** (1989) 641.

[6] P. Christe and G. Mussardo, *Nucl. Phys.* **B330** (1990) 465, *Int. J. Mod. Phys.* **A5** (1990) 4581.

[7] V. A. Fateev and A. B. Zamolodchikov, *Int. J. Mod. Phys.* **A 5** (1990) 1025.

[8] G. Sotkov and C.J. Zhu, *Phys. Lett.* **B229** (1989) 391.

[9] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.* **B338** (1990) 689.

[10] F.A. Smirnov, *Int. J. Mod. Phys.* **A6** (1991) 1407.

[11] A.V. Zhiber and A.B. Shabat, *Sov. Phys. Dokl.* **24** (1979) 607.

[12] A.V. Mikhailov, *P'isma Zh. Eksp. Theor. Fiz.* **30** (1979) 443.

[13] A.G. Izergin and V.E. Korepin, *Commun. Math. Phys.* **79** (1981) 303.

[14] C.N. Yang and C.P. Yang, *Phys. Rev.* **147** (1966) 303; *J. Math. Phys.* **10** (1969) 1115.

[15] Al. B. Zamolodchikov, *Nucl. Phys.* **B342** (1990) 695.

[16] T.R. Klassen and E. Melzer, *Nucl. Phys.* **B350** (1991) 635.

[17] V. A. Fateev, *Phys. Lett.* **B324** (1994) 45.

[18] M. Karowski and P. Weisz, *Nucl. Phys.* **B139** (1978) 445.

[19] F.A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, World Scientific, Singapore, (1992).

[20] M. T. Batchelor and K. A. Seaton, *Eur. Phys. J.* **B 5** (1998) 719.

[21] K. A. Seaton and M. T. Batchelor, in *Group22: Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics*, edited by S. P. Corney, R. Delbourgo and P. D. Jarvis (International Press, Cambridge MA, 1998), pp 274-278.

[22] K. A. Seaton and M. T. Batchelor, math-ph/0110021, to appear in J. Math. Phys. (2001).

[23] V. V. Bazhanov, B. Nienhuis and S. O. Warnaar, *Phys. Lett.* **B 322** (1994) 198.

[24] U. Grimm and B. Nienhuis, in Statistical Models, *Yang-Baxter Equation and Related Topics*, edited by M.L. Ge and F.Y. Wu (World Scientific, Singapore, 1996) pp 384-393.

[25] U. Grimm and B. Nienhuis, *Phys. Rev.* **E55** (1997) 5011.

[26] U. Grimm and B. Nienhuis, private communication.

[27] R. J. Baxter, in *Fundamental Problems in Statistical Mechanics V*, edited by E.G.D. Cohen, North-Holland Publishing Co., Amsterdam (1980)

[28] P. A. Pearce, *Phys. Rev. Lett.* **58** (1987) 1502.

[29] A. Klümper and J. Zittartz, *Z. Phys.* **B71** (1988) 495; *Z. Phys.* **B75** (1989) 371.

[30] P.E. Dorey, *Nucl. Phys.* **B358** (1991) 654; *Nucl. Phys.* **B374** (1992) 741.

[31] A. Fring, H.C. Liao and D.I. Olive, *Phys. Lett.* **B266** (1991) 82.

[32] A. Fring and D.I. Olive, *Nucl. Phys.* **B379** (1992) 429.

[33] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, *Phys. Lett.* **B87** (1979) 389

[34] C. Destri and H.J. De Vega, *Phys. Lett.* **B233** (1989) 336, *Nucl. Phys.* **B358** (1991) 251.

[35] G. Mussardo, *Phys. Rep.* **218** (1993) 215.

[36] B. M. McCoy and W. P. Orrick, *Phys. Lett.* **A230** (1997) 24.

[37] M. T. Batchelor and K. A. Seaton, *J. Phys.* **A30** (1997) L479.

[38] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).

[39] D.I. Olive, *Nuovo Cimento* **26** (1962) 73.

[40] C. Korff and K. A. Seaton, in preparation.

[41] D. Fioravanti, G. Mussardo and P. Simon, *Phys. Rev. Lett.* **85** (2000) 126; *Phys. Rev.* **E63** (2001) 016103.

[42] A. Fring, C. Korff and B.J. Schulz, *Nucl. Phys.* **B567** (2000) 409.

[43] C. Korff, *Lie algebraic structures in integrable models, affine Toda field theory*, doctoral thesis, July 2000, Freie Universität Berlin; http://darwin.inf.fu-berlin.de/2000/99/

[44] M. Caselle and M. Hasenbusch, *Nucl. Phys.* **B579** (2000) 667.

[45] K. A. Seaton and M. Batchelor, *J. Stat. Phys.* **102** (2001) 1019.

[46] R. Kedem, T.R. Klassen, B. M. McCoy and E. Melzer, *Phys. Lett.* **B304** (1993) 263.

[47] S. O. Warnaar and P. A. Pearce, *J. Phys.* **A27** (1994) L891.

[48] S. O. Warnaar and P. A. Pearce, *Int. J. Mod. Phys.* **A11** (1996) 291.

[49] M. Lässig, G. Mussardo and J. L. Cardy, *Nucl. Phys.* **B348** (1991) 591

[50] K. A. Seaton, hep-th/0111144, to appear in *J. Stat. Phys.* (2002)

[51] B. Kostant, *Amer. J. Math.* **81** (1959) 973, *Proc. Natl. Acad. Sci.* USA **81** (1984) 5275.